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THE SIGN-DEFINITE CRITERION OF A HOMOGENEOUS POLYNOMIAL IN A CONE†

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A sign-definite criterion of a polynomial of degree m in a cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ of space R^n is proposed and also a method of investigating these properties based on certain results obtained from Sirazetdinov. This enables the solution of the problem of the stability of systems of differential equations with polynomial right-hand sides to be simplified.

1. FORMULATION OF THE PROBLEM

IN CERTAIN problems of stability (for example, in problems in economics, stability in biological societies, etc.), there is no need to use functions with sign-definite properties over the whole of the space R^n. For the systems of ordinary differential equations which describe these processes, a certain set $K \subset R^n$ is positively invariant. The trajectories of the system with initial data from K do not leave its limits as time passes. This set is called a cone, it is closed and all its elements possess the following properties: (1) for any $x \in K$ it follows that $-x \bar{\in} K$ ($x \neq 0$, 0) is zero, and (2) for any α, $\beta > 0$ and arbitrary u, $v \in K$ it follows that $\alpha u + \alpha v \in K$.

Henceforth we will consider the case when the cone coincides with the coordinate angle. We will use the notation [1] $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$, $\alpha_{i0} \in N_0 = \{-1, 1\}$. Here $\{\alpha_{i0}\}$, $(i = 1, \ldots, n)$ is the basis of the cone K. In this case

$$\alpha_{i0} = \operatorname{sign} x_i \, (x_i \neq 0), \qquad i = 1, \ldots, n; \qquad \alpha_{i0} x_i \geqslant 0$$

If the problem involves considering a system of ordinary differential equations whose trajectories do not leave the limits of the cone as time passes, then when solving the problem of the stability of this system there is no need to use as the function a Lyapunov function that is sign-definite over the whole of space. It is sufficient for it to possess this property solely in the cone K.

Hence, the problem arises of investigating the sign-definite properties of different functions, in particular, homogeneous polynomial-forms in a certain cone K of space R^n.

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2. THE NECESSARY AND SUFFICIENT CONDITIONS FOR A FORM OF A CERTAIN mth ORDER TO BE SIGN-DEFINITE IN A CONE

Consider the following problem. Suppose that in a certain region

$$G = \{x : 0 \leq \| x \| = |x_1| + |x_2| + \ldots + |x_n| < \infty,\quad x \in R^n\}$$

we are given the function

$$W(x) = \sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m}$$

$$a_{i_1 \ldots i_m} = a_{i_m \ldots i_{m-1}} = \ldots = a_{i_2 \ldots i_m i_1} = \text{conts}$$

as a form of the mth order (m is an integer positive number).

We will investigate the sign-definite properties of the function in a cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$.

To fix our ideas, and also in order to simplify the calculations, we will investigate the positive-definite properties W in the first coordinate angle $x \geq 0$. To do this we make the replacement of variables $x_i = u_i^2$ ($i = 1, \ldots, n$) and obtain a certain function $\Omega(u)$ of order $2m$, the sign-definite properties of which in the whole space of the variables $u_1, \ldots, u_n$ are identical with the sign-definite properties of the form $W(x)$ in the cone $x \geq 0$. The functions W and Ω are continuous with respect to the arguments x and u, respectively, and possess the property of homogeneity. Hence, starting from Weierstrass' theorem on the properties of continuous functions, we will only investigate the function $\Omega(u)$ on a sphere S of unit radius. To do this, we construct the Lagrange function

$$L(u) = \Omega(u) + m\lambda(1 - u_1^2 - \ldots - u_n^2)$$

and we investigate it at an absolute extremum. As a result we obtain the following system of equations:

$$\partial L(u)/\partial u_i = \partial\Omega(u)/\partial u_i - 2m\lambda u_i = 0, \quad i = 1, \ldots, n \qquad (2.1)$$
$$u_1^2 + \ldots + u_n^2 = 1$$

Let Q_m be a set of all strictly increasing sequences $\sigma = (i_1, \ldots, i_r)$ [2], made up of r numbers of the set $N = \{1, \ldots, n\}$. Then system (2.1) can be converted to the form

$$2mu_{i_s}\Big(\frac{1}{m}\sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1\ldots i_m} u_{i_1}^2 \ldots u_{i_m}^2 + \ldots + \frac{m-1}{m} u_{i_s}^{2m-4} \sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1\ldots i_m} u_{i_1}^2 \ldots u_{i_m}^2 +$$
$$+ a_{i_s\ldots i_s} u_{i_s}^{2m-2} - \lambda\Big) = 0, \quad i_s = 1, \ldots, n \qquad (2.2)$$
$$i_l \neq i_s \ (l = 1, \ldots, m); \quad u_1^2 + \ldots + u_n^2 = 1$$

In the first n equation of system (2.2) we equate each of the factors to zero, and, taking the connecting equation into account, we obtain a set of systems of non-linear algebraic equations. Moreover, by Euler's theorem on homogeneous functions, we have

$$\Omega(u) = \frac{1}{2m}\sum_{i=1}^{n} \frac{\partial\Omega(u)}{\partial u_i} u_i$$

and since it follows from Eqs (2.1) that $\partial\Omega(u)/\partial u_i = 2m\lambda u_i$, we obtain that at extremum points on the sphere S the function $\Omega(u) = \lambda$.

Hence, the function $\Omega(u)$ is positive-definite if system (2.2) when $\lambda \leq 0$ has no non-zero solutions; the systems of non-linear algebraic equations

$$\sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1\ldots i_m}^{(\sigma)} x_{i_1} \ldots x_{i_m} = \lambda, \quad i_s \in \sigma$$
$$s = 1, \ldots, m; \quad \sigma \in Q_m \qquad (2.3)$$
$$x_\gamma = 0, \quad \gamma = N \backslash \sigma, \quad x_\gamma \in R^{n-r}, \quad r = 1, \ldots, n$$

($a_{i_1\ldots i_m}^{(\sigma)}$ are coefficients of the form $W(x)$ with indices from σ) when $\gamma \leq 0$ should also not have any non-zero solutions which belong as a whole to the cone $x \geq 0$. The converse is also true. For the case of negative definiteness it is necessary to have $\lambda \geq 0$, while in the cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ it is necessary first of all to make the replacement of variables $y_i = \alpha_{i0} x_i$ and repeat the previous reasonings.

Hence, we will formulate the following theorem.

Theorem 1. In order that the form $W(x)$ of order $m \geq 2$ $(m \in Z_+)$ should be positive (negative)-definite in the cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ of the space R^n, it is necessary and sufficient that the systems of non-linear algebraic equations

$$\sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i_1 \ldots i_m}^{(\sigma)} x_{i_1} \ldots x_{i_m} = \alpha_{i_s 0} \lambda$$

$$i_s \in \sigma, \quad s = 1, \ldots, m; \quad \sigma \in Q_{rn}$$

$$x_\gamma = 0, \quad \gamma = N \backslash \sigma, \quad x_\gamma \in R^{n-r}, \quad r = 1, \ldots, n \tag{2.4}$$

when $\lambda \leq 0$ $(\lambda \geq 0)$ should have no non-zero solutions which belong as a whole to the cone.

3. THE SIGN-DEFINITE CRITERION. THE PLANE CASE

Since $\Omega(u)$ is a form of even order, we can use the results obtained in [5]. We carry out the transformation

$$z_1 = u_1^m, \quad z_2 = u_1^{m-1} u_2, \ldots, z_l = u_n^m$$

result of which the function Ω is converted into a quadratic form

$$V(z) = \sum_{i=1}^{l} \sum_{j=1}^{l} v_{ij} z_i z_j, \quad v_{ij} = v_{ji} = \text{const}$$

It is obvious that the vectors $z = \text{col}(z_1, \ldots, z_l)$ are linearly independent and form a space of dimensions l, which we will denote by Z_l. Hence,

$$\forall u \in G^* \subseteq R^n, \qquad \exists z \in Z_l: \Omega(u) = V(z)$$

and in this case the sign-definite properties of $V(z)$ imply that the function $W(x)$ is sign-definite in the cone K. The problem of investigating W can be reduced to a simpler one.

Note that the approach described only gives sufficient conditions for sign-definiteness. For example, if $W(x)$ is a third-order form

$$W(x) = a_{30} x_1^3 + a_{21} x_1^2 x_2 + a_{12} x_1 x_2^2 + a_{03} x_2^3$$

$x = \text{col}(x_1, x_2)$, then using Theorem 1 we can obtain that for positive (negative) definiteness in a certain cone $K\{\alpha_{10}, \alpha_{20}\}$ of the space R^2 it is necessary and sufficient that in this region the systems of non-linear algebraic equations

$$\begin{cases} a_{30} x_1^2 + {}^2/_3 a_{21} x_1 x_2 + {}^1/_3 a_{12} x_2^2 = \alpha_{10} \lambda \\ {}^1/_3 a_{21} x_1^2 + {}^2/_3 a_{12} x_1 x_2 + a_{03} x_2^2 = \alpha_{20} \lambda \end{cases}$$

$$\begin{cases} x_1 = 0 \\ a_{03} x_2^2 = \alpha_{20} \lambda \end{cases}, \qquad \begin{cases} x_2 = 0 \\ a_{30} x_1^2 = \alpha_{10} \lambda \end{cases}$$

[(a_{ml} = const are coefficients of the form $W(x)$] should have no non-zero solutions for all $\lambda \leq 0$ $(\lambda \geq 0)$. Using the above method, the similar conditions, for example, in the first quadrant, reduce to positiveness (negativeness) of all the coefficients of the form W.

We can similarly formulate the conditions for a fourth-order form to be sign-definite.

Note that the conditions for a fourth-order form of two variables to be sign-definite everywhere in R^2 were obtained in [6] and elsewhere. However, as can be seen, in a number of problems it is not always essential that the function should possess the same properties over the whole of the space of the variables.

Consider the following example. Suppose we are given the function

$$g(x, y) = \frac{1}{32} x^4 + x y^3 + y^4$$

It is sign-variable over the whole space R^2 of the variables x and y and correspondingly positive-definite in the first coordinate angle. Moreover, the corresponding subsystems

$$\begin{cases} \frac{1}{32}x^3 + \frac{1}{4}xy^2 = \lambda \\ \frac{3}{4}xy^2 + y^3 = \lambda \end{cases}, \quad \begin{cases} x = 0 \\ y^3 = \lambda \end{cases}, \quad \begin{cases} y = 0 \\ x^3 = \lambda \end{cases}$$

when $\lambda \leqslant 0$ do not have non-zero solutions which belong as a whole to the cone $x \geqslant 0$, $y \geqslant 0$.

We can formulate the sign-definite conditions of a form of order m of two variables in a certain cone $K\{\alpha_{10}, \alpha_{20}\}$ of the space R^2 as follows.

For a form of a certain mth order

$$\Phi(x_1, x_2) = A_{m0}x_1^m + A_{m-1,1}x_1^{m-1}x_2 + \ldots + A_{0m}x_2^m$$

to be positive (negative)-definite in the cone $K\{\alpha_{10}, \alpha_{20}\}$ of space R^2 it is necessary and sufficient that the system of non-linear algebraic equations

$$\begin{cases} A_{m0}x_1^{m-1} + \frac{m-1}{m}A_{m-1,1}x_1^{m-2}x_2 + \ldots + \frac{m-l}{m}A_{m-l,l}x_1^{m-l-1}x_2^l + \ldots \\ \ldots + \frac{1}{m}A_{1,m-1}x_2^{m-1} = \alpha_{10}\lambda \\ \frac{1}{m}A_{m-1,1}x_1^{m-1} + \frac{2}{m}A_{m-2,2}x_1^{m-2}x_2 + \ldots + \frac{l}{m}A_{m-l,l}x_1^{m-l}x_2^l + \ldots + A_{0m}x_2^{m-1} = \alpha_{20}\lambda \end{cases}$$

$$\begin{cases} x_1 = 0 \\ A_{0m}x_2^{m-1} = \alpha_{20}\lambda \end{cases}, \quad \begin{cases} x_2 = 0 \\ A_{m0}x_1^{m-1} = \alpha_{10}\lambda \end{cases}$$

should have no non-zero solutions in the region considered for all $\lambda \leqslant 0$ ($\lambda \geqslant 0$).

In many cases when solving the problem of the stability of systems of differential equations it turns out to be more useful and effective to use a higher-order form with sign-definite properties in a certain cone $K \subset R^n$ instead of the quadratic forms usually employed. The method itself is then simplified considerably.

4. MONOTONIC STABILITY OF ONE TYPE OF SYSTEMS OF DIFFERENTIAL EQUATIONS

The results obtained above simplify the problem of investigating the properties of monotonic stability in a certain cone $K \subset R^n$ of one type of system of differential equations with polynomial right-hand sides of special form. Here we have in mind systems in which the right-hand sides are homogeneous polynomials of a certain degree.

Consider in a certain region

$$G = \{(t, x):\ t \geqslant t_0\quad x \in R^n,\ \ 0 < \|x\| = \sqrt{(x, x)} < \infty\}$$

the system of ordinary differential equations (m is a certain integer)

$$\dot{x}_s = -x_s R_s^{(m)}(x), \quad s = 1, \ldots, n \qquad (4.1)$$

$$R_s^{(m)}(x) = \sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} \beta^s_{i_1 \ldots i_m} x_{i_1 \ldots i_m}$$

Definition. System (4.1) is monotonically stable in the cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ of the space R^n if the function

$$S(t) = \sum_{i=1}^{n} \alpha_{i0}x_i(t)$$

strictly monotonically decreases in a time t along trajectories of system (4.1) with initial data from the cone.

Hence, we can formulate the following result on the basis of this definition.

Theorem 2. For monotonic stability of system (4.1) in a certain cone $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ of the space R^n it is sufficient that in this cone the form of order $m+1$

$$W(x)=\sum_{s=1}^{n}\alpha_{s0}x_s\sum_{i_1=1}^{n}\ldots\sum_{i_m=1}^{n}\beta^s_{i_1\ldots i_m}x_{i_1\ldots i_m}$$

should be positive-definite. Here $\beta^s_{i_1\ldots i_m}$ = const are the coefficients of system (4.1).

It is important to note that when the conditions of Theorem 2 are satisfied, the monotonic stability of system (4.1) in the part of the space R^n considered is equivalent to its asymptotic stability.

Using the definition of monotonic stability one can also formulate a number of other conditions for the monotonic stability of system (4.1) in a certain cone K which arise from different ratios of the coefficients $\beta^s_{i_1\ldots i_m}$ and the elements of the basis of this cone. For example, we have the following assertion: suppose the coefficients of system (4.1) and the elements of the basis of a certain cone $K\{\alpha_{10},\ldots,\alpha_{n0}\}$ are connected by the relation

$$\sum_{s=1}^{n}\beta^s_{i_1\ldots i_m}\alpha_{s0}=0\quad(s\neq i_l,\ l=1,\ldots,m)$$

and $\alpha_{s0}\beta^s_{s\ldots s}>0$, if m is even and $\beta^s_{s\ldots s}>0$ if m is odd. Then system (4.1) is monotonically stable in the cone $K\{\alpha_{10},\ldots,\alpha_{n0}\}$ of the space R^n.

To prove the assertion it is sufficient to multiply each sth equation of the system by α_{s0} and sum them over s from 1 to n.

We see from the above that the properties of the stability of system in a cone can be investigated by checking that the form constructed from the coefficients of the system and the elements of the basis of the cone considered are sign-definite.

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